Lectures on Game Comonads in Finite Model Theory

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MATH 6710: Category Theory, Categorical Logic, and Type Theory

Contents

1 Lecture I

1.1 Introduction and Motivation

These lectures are about a recent development inroduced by Abramsky, Dawar, and Wang introduced in 2017 providing a novel connection between category theory and (finite) model theory via a construction called game comonads (1). In (2), Abramsky introduces the topic as bridging a fundamental divide between two areas of theoretical computer science - one which focuses on semantics and formal methods (structure) and the other focusing on algorithms and complexity (power). Semantics is often studied using category theory, and complexity is often studied using finite model theory.

A central tool in finite model theory for studying the expressive power of logics is the technique of *model comparison games*. Model comparison games are a method to determine elementarily equivalence between two first order structures, and their main application is in proving undefinability of certain properties in first-order logic. Game comonads provide a categorical semantics for model comparison games by formulating them as comonads, effectively relating "structure" and "power".

In these lectures we will focus on a particular model comparison game, called the Ehrenfeucht-Fraisse (EF) game. We will show how EF games can be modeled as comonads, and show that there is a bijective correspondence between morphisms in the resulting coKleisli category, and winning strategies in the EF game. We then will show a combinatorial result using coalgebras, namely that the coalgebra number (to be defined) of a structure $\mathscr{A}$ for the EF comonad corresponds to the tree depth of $\mathscr{A}$. We will then build up some new machinery in category theory to discuss an elegant categorical generalization of Lovazs' theorem. Lovasz' theorem states that for any two finite relational structures A and B, $A \cong B \iff \forall C,\ |\mathrm{Hom}(C, A)| = |\mathrm{Hom}(C, B)|$ (5). We will prove that this property, which is called *combinatoriality*, holds for any locally finite category with pushouts and a *proper factorization system.*

1.2 Finite Model Theory

Finite model theory is an area of logic which branched off into an area of theoretical computer science. Finite model theory is what it sounds like - it is the theory of those first-order structures whose universes are finite.

Finite model theory turns out to be a pretty distinct field from classical model theory, simply due to the fact that many of the fundamental results of first-order model theory don't hold when restricted to finite domains. Namely, the compactness theorem and Godel's completeness theorem fail when restricted to finite models. If you've ever taken a logic class you may remember frequently using the compactness theorem in your proofs. These results lead to FMT looking very distinct to classical model theory in terms of its methods and results.

1.2.1 Why Finite?

What makes finite model theory interesting is the fact that in computer science, all the structures of interest will be over a finite universe, since computers have finite memory. So FMT gives us a way to study the expressive power of logics over computational structures, thus allowing us to explore things like definability in a computational setting.

Another tool which FMT equips us with is the ability to use logic to examine the computational complexity of finite structures - leading to the field of descriptive complexity theory. The famous result from descriptive complexity is Fagin's theorem, which states that the class NP is equal to the class of queries which can be expressed in existential second order logic. An immediate consequence of this is that $\mathrm{NP} = \text{co-NP} \iff \exists\mathrm{SO} = \forall\mathrm{SO}$

1.2.2 Review: Languages, Structures, Isomorphisms

We review here some basic ideas in logic.

Definition 1. *A language Ω is a collection of function symbols, relation symbols, and constant symbols, each with a specific arity*

Example 1. *The language of groups is the set $\{\cdot, ^{-1}, e\}$*

Definition 2. *A relational vocabulary σ is a language with no function symbols*

Example 2. *The language of graphs $\{E\}$ where E is a binary relation symbol, is a relational vocabulary*

Central to model theory is the *interpretation* of languages, which we call structures.

Definition 3. *Given a language σ, a σ-structure $\mathscr{M}$ consists of a set M called the universe with the following:*

- *For each function symbol f of arity n, a function $f^{\mathscr{M}} : M^n \to M$*
- *For each relation symbol R of arity n a relation $R^{\mathscr{M}} \subseteq M^n$*
- *For each constant symbol c, an element $c^{\mathscr{M}} \in M$*

Example 3. *A group G is simply a σ-structure (where σ is the language of groups) which satisfies the group axioms*

Given some structures, we want to be able to define the notion of a structure-preserving map between them. We call these homomorphisms and define them as follows:

Definition 4. *Given two σ-structures $\mathscr{A}, \mathscr{B}$, a function $h : A \to B$ is a homomorphism iff:*

- *For each relation symbol R, if $(a_1, .., a_n) \in R^{\mathscr{A}}$ then $(h(a_1), .., h(a_n) \in R^{\mathscr{B}}$*
- *For each function symbol f, $f(h(a_1), ..., h(a_2)) = h(f(a_1, .., a_n))$*
- $h(c^{\mathscr{A}}) = c^{\mathscr{B}}$

Definition 5. *A bijective homomorphism is an isomorphism*

Definition 6. *A partial isomorphism between structures $\mathscr{A}$ and $\mathscr{B}$ is a function f such that:*

- $dom(f) \subset A$
- $rng(f) \subset B$
- *f is an isomorphism on the susbtructures generated by its domain and range*

1.3 Ehrenfeucht-Fraisse Games

Our object of study is the Ehrenfeucht-Fraisse (EF) game and its comonadic semantics. As I mentioned previously, EF games are a tool for determining elementary equivalence between structures, and thus can be used to prove the undefiniability of certain structural properties. The k-round EF game is played on two structures A and B over the same relational vocabulary σ between two players Spoiler and Duplicator. The Duplicator aims to show that the two structures are elementarily equivalent, and Spoiler aims to show they are not.

In a single round, Spoiler selects an element from either A or B. Duplicator then selects an element in the other structure. Intuitively, the Duplicator is always looking to pick a "similar" element in the opposite structure as the Spoiler. If, after k rounds, there is an isomorphism between the substructures generated by the chosen elements, we say Duplicator has won the game.

Notation: If there exists a duplicator winning strategy for the k-round EF game on A and B, we say $A \sim_k B$

Theorem 1 (Ehrenfeucht-Fraisse Theorem)**.** *$A \sim_k B$ for all $k \in \mathbb{N}$ if and only if $A \equiv B$*

The proof can be found in (4). From this one can derive the following:

Corollary 1. *Let Q be a property of structures (eg. graph acyclicity). If we have that:*
- $A \sim_k B$, for all $k \in \mathbb{N}$ however
- $Q(A) \neq Q(B)$
Then Q is not FO-definable

1.3.1 Existential EF Games

For our purposes, we will focus on a particular type of EF game, called the *existential* EF game. The premise is similar, but the idea is that Spoiler can only select elements in A, and duplicator can only select elements in B. The Duplicator wins the *existential* k-round EF game iff after k rounds, there is a partial *homomorphism* between the two generated substructures. This gives us an existential positive version of the EF theorem.

Notation: We say $A \equiv^{\exists+} B$ when A agrees with B on the existential positive fragment of FOL - ie. no universal quantifiers or negation. Further, $A \sim_k^{\exists} B$ when Duplicator has a winning strategy in the k-round existential EF game.

Theorem 2 (Existential EF Theorem)**.** *$A \sim_k^{\exists} B$ for all $k \in \mathbb{N}$ if and only if $A \equiv^{\exists+} B$*

1.4 Category Theory

1.4.1 The Category Σ

Our setting is the category Σ, where objects are structures over the relational vocabulary σ, and morphisms are σ-structure homomorphisms. Similarly one can talk about Σ_f, the category of finite σ-structures.

1.4.2 Comonads

We review comonads and comonads provide an alternative (and more convenient) definition in *Kleisli form*. Recall that a comonad (G, ϵ, δ) over a category $\mathbb{C}$ is a triple consisting of:

- An endofunctor $G : \mathbb{C} \to \mathbb{C}$
- A natural transformation $\epsilon : G \to \mathbf{1}$ (counit)
- A natural transformation $\delta : G \to G^2$ (comultiplication)

Such that the following diagrams commute:

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "G\delta"] \\ G^2 \arrow[r, "\delta G"'] & G^3 \end{tikzcd} \qquad \begin{tikzcd} G^2 \arrow[r, "G\epsilon"'] \arrow[d, "\epsilon G"] \arrow[dr, dash, "\mathrm{id}" description] & G \arrow[d, "\delta"'] \\ G \arrow[r, "\delta"] & G^2 \end{tikzcd}$$

$$\delta_G \circ \delta = G\delta \circ \delta$$

$$\epsilon_G \circ \delta = \mathrm{id}_G$$
$$G\epsilon \circ \delta = \mathrm{id}_G$$

These are called coassociativity and counit laws (dual to monad laws).

An equivalent formulation is a *comonad in Kleisli* form, given by an object map G (note: not a functor!), morphisms $\epsilon_A : GA \to A$ for each object A, and a *coextension* operation $()^*$ taking each morphism $f : GA \to B$ to a morphism $f^* : GA \to GB$. These must satisfy the following equations:

$$\epsilon_A^* = id_{GA}, \ \epsilon \circ f^* = f, \ (g \circ f^*)^* = g^* \circ f^*$$

We can derive a comonad from a comonad in Kleisli form. We extend G to a functor by $Gf = (f \circ \epsilon)^*$.

We then define the comultiplication $\delta_A : GA \to G^2A$ by $\delta_A = id_{GA}^*$

Conversely, given a comonad (G, ϵ, δ), we define the coextension by $f^* = Gf \circ \delta_A$.

1.4.3 coKleisli Category

The reason we use this Kleisli form is because it allows us to give a simple definition of a comonad's corresponding coKleisli category. Given a comonad G in coKleisli form on a category $\mathbb{C}$, we define the coKleisli category $\mathbf{Kl}(G)$ for G as follows:

- $\mathrm{Ob}(\mathbf{Kl}(G)) = \mathrm{Ob}(\mathbb{C})$
- Morphisms $A \to B$ are given by the morphisms $GA \to B$
- Composition of Kleisli arrows $f : GA \to B, g : GB \to C$ given by $g \cdot f = g \circ f^*$

1.5 The EF Comonad

In this section we define the comonadic formulation of the existential EF game. Note we're working in the category Σ of σ-structures. Most of the results and definitions from this section come from (2)

We define the k-round existential EF game comonad $\mathbb{E}_k$ as follows:

Given a σ-structure $A \in \mathsf{Ob}(\Sigma)$, we define a new structure $\mathbb{E}_k(A)$ whose universe consists of all sequences of length at most k of elements of A, ie

$$A \mapsto A^{\leq k}$$

These sequences correspond to spoiler plays - the sequence of elements spoiler has picked from A after $i \leq k$ rounds

The counit $\epsilon_A : A^{\leq k} \to A$ projects the last element of a sequence, ie:

$$[a_1, ..., a_j] \mapsto a_j$$

For each relation symbol R of arity n, $R^{\mathbb{E}_k(A)}$ is the set of n-tuples $(s_1, .., s_n)$ such that:

- each s_i, s_j is comparable in the prefix ordering ($s_i \uparrow s_j$)
- $R^A(\epsilon_A s_1, .., \epsilon_A s_n)$

And, given $f : \mathbb{E}_k A \to B$, the coextension $f^* : \mathbb{E}_k A \to \mathbb{E}_k B$ as

$$[a_1, ..., a_j] \mapsto [b_1, .., b_j]$$

such that each $b_i = f[a_1, .., a_i]$

Proposition 1. *$(\mathbb{E}_k, \epsilon, (\cdot)^*)$ is a comonad on Σ*

Proof. $\varepsilon_A^* = \mathrm{id}_{E_k A}$.

For any sequence $[a_1, \ldots, a_j] \in A^{\leq k}$, we have

$$\varepsilon_A^*([a_1, \ldots, a_j]) = [b_1, \ldots, b_j],$$

where $b_i = \varepsilon_A([a_1, \ldots, a_i]) = a_i$. Therefore,

$$\varepsilon_A^*([a_1, \ldots, a_j]) = [a_1, \ldots, a_j] = \mathrm{id}_{E_k A}([a_1, \ldots, a_j]).$$

Conclude that, $\varepsilon_A^* = \mathrm{id}_{E_k A}$.

$\varepsilon_B \circ f^* = f$.

For any sequence $[a_1, \ldots, a_j] \in A^{\leq k}$, we compute

$$(\varepsilon_B \circ f^*)([a_1, \ldots, a_j]) = \varepsilon_B(f^*([a_1, \ldots, a_j])) = \varepsilon_B([b_1, \ldots, b_j]) = b_j,$$

where $b_i = f([a_1, \ldots, a_i])$. Specifically, $b_j = f([a_1, \ldots, a_j])$, so

$$(\varepsilon_B \circ f^*)([a_1, \ldots, a_j]) = f([a_1, \ldots, a_j]).$$

Conclude that $\varepsilon_B \circ f^* = f$.

$(g \circ f^*)^* = g^* \circ f^*$.

Let $f : E_k A \to B$ and $g : E_k B \to C$ be homomorphisms. For any sequence $[a_1, \ldots, a_j] \in A^{\leq k}$, we first compute $f^*([a_1, \ldots, a_i])$ for $1 \leq i \leq j$:

$$f^*([a_1, \ldots, a_i]) = [b_1, \ldots, b_i],$$

where $b_t = f([a_1, \ldots, a_t])$ for $1 \leq t \leq i$.

Next, we compute $(g \circ f^*)([a_1, \ldots, a_i])$:

$$(g \circ f^*)([a_1, \ldots, a_i]) = g(f^*([a_1, \ldots, a_i])) = g([b_1, \ldots, b_i]) = c_i.$$

Then, $(g \circ f^*)^*([a_1, \ldots, a_j]) = [c_1, \ldots, c_j]$, where $c_i = g([b_1, \ldots, b_i])$.

On the other hand, computing $g^*(f^*([a_1, \ldots, a_j]))$, we have:

$$g^*(f^*([a_1, \ldots, a_j])) = g^*([b_1, \ldots, b_j]) = [c'_1, \ldots, c'_j],$$

where $c'_i = g([b_1, \ldots, b_i])$.

Since $c_i = c'_i$ for all $1 \leq i \leq j$, it follows that

$$(g \circ f^*)^*([a_1, \ldots, a_j]) = g^*(f^*([a_1, \ldots, a_j])).$$

Therefore, $(g \circ f^*)^* = g^* \circ f^*$. $\square$

Theorem 3. *TFAE:*

1. *There exists a morphism $\mathbb{E}_k A \to B$*
2. *Duplicator has a winning strategy in the k-round existential EF game*

Proof.

$\Longrightarrow$: *Suppose Spoiler plays $s = [a_1, ..., a_k]$. Then we define the duplicator's strategy as follows:*
For each $i \in \{1, .., k\}$, if for some $j < i$, $a_j = a_i$ (ie. Spoiler chose duplicates), then $b_i = b_j$, (ie. Duplicators i^{th} pick is their previous j^{th} pick). Otherwise, $b_i = f[a_1, .., a_i]$.

It remains to show that $\gamma = \{(a_i, b_i) \mid 1 \leq i \leq k\}$ is a partial homomorphism. By construction, if $a_i = a_j$ then $b_i = b_j$, so the function is well-defined. To show homomorphism, suppose $R \in \sigma$ and suppose $R^A(a_{i_1}, .., a_{i_n})$ with $i_j \in \{1, .., k\}$ for all $j \leq n$. Let s_{i_j} denote the initial segment ending in a_{i_j}. Since each s_{i_j} is an initial segment of s, they are pairwise prefix comparable, and by definition, $\epsilon(s_{i_j}) = a_{i_j}$. Recalling the definition of $R^{\mathbb{E}_k}$, it follows immediately that $R^{\mathbb{E}_k}(s_{i_1}, .., si_n)$, and since f is a homomorphism, $R^B(b_{i_1}, ..., b_{i_n})$

$\Longleftarrow$: *Suppose Duplicator has a winning strategy in the k-round existential game. Then by definition, for any sequence of rounds $s = [a_1, ..., a_k]$ in A, there exists some sequence $t = [b_1, ..., b_k]$ in B such that $\gamma := a_i \mapsto b_i$ is a partial homomorphism. Given an arbitrary play s, let γ_s be the resulting partial homomorphism taking each Spoiler pick to the Duplicator response. Then we define $f : \mathbb{E}_k A \to B$ as $f(s_i) = \gamma_s(a_i)$ where s_i is a prefix of s ending in a_i.*

We claim that since each γ_s is a partial homomorphism, f is a homomorphism. Given an arbitrary relation symbol $R \in \sigma$, suppose $R^{\mathbb{E}_k}(s_1, ..., s_m)$ for sequences $s_1, ...s_m$. By the pairwise comparability condition on $R^{\mathbb{E}_k}$, there is some greatest $s = [a_1, ..., a_l]$ wrt prefix amongst the s_i. Since $s_i \sqsubseteq s$, then the last move of s_i, aka $\epsilon(s_i)$, is in $\{a_1, ..., a_l\}$. Then by the homomorphism property of ϵ, $R^A(\epsilon(s_1), ..., \epsilon(s_m))$. Since γ_s is a a partial homomorphism, then $R^B(\gamma\epsilon(s_1), ..., \gamma\epsilon(s_m))$, and since $f(s_i) = \gamma_s(a_i)$, then $R^B(f(s_1), ..., f(s_1))$ $\square$

2 Lecture II

2.1 Recap

In the previous lecture, we've defined the existential EF comonad and showed that coKleisli morphisms correspond biconditionally to winning duplicator strategies. We recall the definition of the comonad $\mathbb{E}_k$ below.

- The object map is defined as $A \mapsto A^{\leq k}$
- The counit $\epsilon_A : A^{\leq k} \to A := [a_1, .., a_j] \mapsto a_j$ projects the last element of a sequence.
- The relational structure of an object $\mathbb{E}_k A$ is defined such that for any $R \in \sigma$, of arity n $R^{E_k A}$ consists of all tuples $(s_1, ..., s_n)$ such that
  - $s_i \uparrow s_j$ for each $i, j \in [1..k]$
  - $R^A(\epsilon(s_1), .., \epsilon(s_n))$
- The coextension $(\cdot)^*$ defined as $f^*(a_1, ..., a_j) = [b_1, .., b_j]$ where $b_i = f(a_1, .., a_i)$
- The functor action taking morphisms $f : A \to B$ to morphisms $\mathbb{E}_k f : \mathbb{E}_k A \to \mathbb{E}_k B := (f \circ \epsilon)^*$
- The comultiplication $\delta_A : \mathbb{E}_k A \to \mathbb{E}_k \mathbb{E}_k A := \mathrm{id}^*_{\mathbb{E}_k A}$

The final theorem from last lecture, combined with the existential EF theorem tells us that TFAE:

- There exists a coKleisli morphism $f : \mathbb{E}_k A \to B$ for all k
- $A \sim^{\exists}_k B$ for all k
- $A \equiv^{\exists +} B$

2.2 Coalgebras

If you've seen monads before you're likely familiar with the associated notion of an algebra over a monad. Similarly, comonads are associated with the notion of a coalgebra.

Definition 7. *A coalgebra over a comonad (G, ϵ, δ) is a morphism $\alpha : A \to GA$ such that the following diagrams commute:*

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \arrow[dr, "id"] & \\ GA \arrow[r, "\epsilon"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & GA \arrow[d, "G\alpha"] \\ GA \arrow[r, "\delta_A"'] & G^2A \end{tikzcd}$$

Since our comonads are indexed as $\mathbb{E}_k$, we can define the following:

Definition 8. *The coalgebra number $\kappa^{\mathbb{E}}(A)$ of a structure A is the least k for which there exists an $\mathbb{E}_k$ coalgebra on A*

We will see that the coalgebra number of a structure is in fact a significant combinatorial measure of the structure.

2.3 Tree Depth in the EF Comonad

For this section we will need some notation and terminology regarding posets.

- A chain in a poset $(P, \leq)$ is a subset $C \subseteq P$ such that for all $x, y \in C, x \uparrow y$
- A forest is a poset $(F, \leq)$ such that for all $x \in F$, the set $\downarrow x = \{y \in F \mid y \leq x\}$ is a finite chain.
- The height $\mathrm{ht}(F)$ of a forest F is $\sup_C |C|$ where C ranges over chains in F

Consider some graph $G = (V, \sim)$ where V is the set of vertices and $\sim$ is the (irreflexive, symmetric) edge relation on V. A forest cover of G is a forest $(F, \leq)$ where $F \subseteq V$ and for any $v_1, v_2 \in V$, $v_1 \sim v_2 \implies v_1 \uparrow v_2$

Definition 9. *The tree depth $td(G)$ of G is $min_F ht(F)$ where F ranges over forest covers of G*

Definition 10. *Given a structure A, the Gaifman graph $\mathscr{G}(A)$ of A is $(A, \sim)$ where $a \sim a'$ iff there exists some relation symbol $R \in \sigma$ and $(a_1, ..., a_k) \in R^A$ such that $a = a_i, a' = a_j$ for some $1 \leq i, j \leq k$*

Proposition 2. *Let $A \in \mathsf{Ob}(\Sigma)$ and $k > 0$. Then there is a bijective correspondence between:*
1. *$\mathbb{E}_k$-coalgebras $\alpha : A \to \mathbb{E}_k A$,*
2. *Forest covers of $\mathscr{G}(A)$ of height $\leq k$.*

Proof. *Suppose $\alpha : A \to \mathbb{E}_k A$ is a coalgebra. For any $a \in A$, let $\alpha(a) = [a_1, \ldots, a_j]$ with $j \leq k$. The counit equation $\epsilon_A \circ \alpha = id_A$ implies that $a_j = a$.*
By definition, the comultiplication δ_A on $\mathbb{E}_k A$ is given by $\delta_A := id^*_{\mathbb{E}_k A}$. Thus,

$$\delta_A(\alpha(a)) = [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_j]].$$

To satisfy the comultiplication equation, $\delta_A \circ \alpha = \mathbb{E}_k(\alpha) \circ \alpha$, we calculate:

$$\mathbb{E}_k(\alpha) \circ \alpha(a) = (\alpha \circ \epsilon)^*([a_1, \ldots, a_j]) = [\alpha(a_1), \ldots, \alpha(a_j)].$$

For equality with $\delta_A(\alpha(a))$ to hold, we must have $\alpha(a_i) = [a_1, \ldots, a_i]$ for each $1 \leq i \leq j$. Thus, α maps each element a in A to an initial segment ending at a, making α injective with a prefix-closed image in $A^{\leq k}$.

Define an order $a \leq a'$ on A by $\alpha(a) \sqsubseteq \alpha(a')$. This order forms a forest cover of $\mathscr{G}(A)$ of height $\leq k$: if $a \smile a'$ in $\mathscr{G}(A)$, then there exists a relation $R^A(a_1, \ldots, a_n)$ with $a = a_i$ and $a' = a_j$ for some i and j. Since α is a homomorphism, $R^{\mathbb{E}_k A}(\alpha(a_1), \ldots, \alpha(a_n))$ holds, and by the pairwise comparability condition on $R^{\mathbb{E}_k A}$, $\alpha(a_i) \uparrow \alpha(a_j)$, so $a_i \uparrow a_j$. Therefore, $(A, \leq)$ is a forest cover of $\mathscr{G}(A)$, of height $\leq k$.

Conversely, given a forest cover $(A, \leq)$ of $\mathscr{G}(A)$ with height $\leq k$, for each $a \in A$, let $a_1 < \cdots < a_j$ be the predecessors of a, with $a_j = a$ and $j \leq k$. Define $\alpha(a) = [a_1, \ldots, a_j]$. Then $\epsilon_A \circ \alpha = id_A$ by construction, and the comultiplication equation holds by a similar argument as above. Since $(A, \leq)$ is a forest cover, if $R^A(a_1, \ldots, a_n)$ holds, we must have $a_i \uparrow a_j$ for all i, j, which implies $R^{\mathbb{E}_k A}(\alpha(a_1), \ldots, \alpha(a_n))$ and ensures that α is a homomorphism.

Theorem 4. *For all $A \in \mathsf{Ob}(\Sigma_f)$, $td(A) = \kappa^{\mathbb{E}}(A)$.*
By Theorem 6.1, for all $k > 0$, $td(A) \leq k$ if and only if $\kappa^{\mathbb{E}}(A) \leq k$.

3 Lecture III

3.1 Lovasz' Theorem

Lovasz' theorem (5) from 1967 states that two finite relational structures A and B are isomorphic if and only if for every finite relational structure C, the number of homomorphisms from C to A is equal to the number of homomorphisms from C to B. One direction of this equivalence is clear, but the other direction requires establishing that the isomorphism type of a structure is characterized entirely by its hom-set.

3.2 Combinatorial Categories

Lovasz' theorem simply states that the category of finite relational σ-structures is *combinatorial*:

Definition 11. *A locally finite category $\mathfrak{C}$ is said to be combinatorial iff*

$$\forall m, n \in \mathsf{Ob}(\mathfrak{C}), m \cong n \iff |Hom(k, m)| = |Hom(k, n)| \ \forall k \in \mathsf{Ob}(\mathfrak{C})$$

3.3 A Categorical Generalization

In this section we will prove the following generalization of Lovazs' theorem:

Theorem 5. *Let A be a locally finite category. If A has pushouts and a proper factorization system, then A is combinatorial. (3)*

3.3.1 Weak Factorization Systems

When dealing with set-functions in every day math, it can be shown that every function can be written as the composition of a surjective function followed by an injective function. Factorizations are a generalization of this idea. We start by defining a weak factorization system.

Definition 12. *Given morphisms e, m in a category A, we say e is left orthogonal m, or m is right orthogonal to e ($e \pitchfork m$), iff for every u, v such that the following diagram commutes:*

$$\begin{tikzcd} \bullet \arrow[r, "e"'] \arrow[d, "u"] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

there exists a diagonal filler morphism w such that the following diagram also commutes:

$$\begin{tikzcd} \bullet \arrow[r, "e"'] \arrow[d, "u"] & \bullet \arrow[d, "v"] \arrow[dl, "w"] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

Notation: Given a class of morphisms $F \in A$, we use $F^{\pitchfork}$ to denote the class of morphisms which are right orthogonal to every morphism in F, ie. the set $\{m \mid \forall e \in F, e \pitchfork m\}$
Similarly for ${}^{\pitchfork}F$, $\{e \mid \forall m \in F, e \pitchfork m\}$

Definition 13. *A pair of classes of morphisms $(\mathscr{E}, \mathscr{M})$ is a weak factorization system iff it satisfied the following conditions:*

1. *Every morphism f in A can be written as $m \circ e$ where $m \in \mathscr{M}$ and $e \in \mathscr{E}$.*
2. *$\mathscr{E} = ^{\pitchfork} \mathscr{M}$ and $\mathscr{M} = \mathscr{E}^{\pitchfork}$*

Definition 14. *A proper factorization system is a weak factorization system where every morphism in $\mathscr{E}$ is an epimorphism and every morphism in $\mathscr{M}$ is a monomorphism*

3.3.2 Pushouts and Proper Factorization Systems in Σ

Given two homomorphisms $f : A \to B$ and $g : A \to C$ in the category Σ of σ-structures, we can construct the pushout by taking the pushout of f and g in the category of sets, which is the disjoint union $D = (B \sqcup C)/\sim$ where $\sim$ is the smallest equivalence relation such that $f(a) \sim g(a)$ for every $a \in A$. We then define f' as $\pi \circ i_l$ and g' as $\pi \circ i_r$, where π is the canonical surjection into an elements equivalence class. Finally we just give $(B \sqcup C)/\sim$ the smallest relational structure making $\pi \circ i_l$ and $\pi \circ i_r$ homomorphisms.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\pi \circ i_l"] \\ C \arrow[r, "\pi \circ i_r"'] & (B \sqcup C)/\sim \end{tikzcd}$$

In a similar way one can also show that Σ is cocomplete, meaning it has all colimits, including these pushouts (exercise). This construction ensures that the pushout of two finite σ-structures remains finite, so it follows that Σ_f has pushouts.

In the category Σ_f of finite σ-structures, we can define a proper factorization system $(\mathscr{E}, \mathscr{M})$ as follows:

$$\mathscr{E} = \{\text{surjective homomorphisms}\}, \quad \mathscr{M} = \{\text{strong/induced embeddings}\}.$$

To demonstrate that this forms a factorization system, consider any homomorphism $f : A \to B$ between σ-structures. We can factor f as a composition

$$A \xrightarrow{f} \tilde{A} \xrightarrow{\iota} B,$$

where $\tilde{A} = f(A) \subseteq B$ is the set-theoretic image of A under f. This factorization involves two maps: 1. The first map $A \to \tilde{A}$ is surjective, making it an $\mathscr{E}$-morphism. 2. The second map $\tilde{A} \to B$ is an embedding, making it a $\mathscr{M}$-morphism. To show it is an embedding we need to define the relational structure of $\tilde{A}$

Relational Structure on $\tilde{A}$. We equip $\tilde{A}$ with a σ-structure by taking each relation $R \in \sigma$ of arity n and defining

$$R^{\tilde{A}} := R^B \cap \tilde{A}^n.$$

This construction restricts the relations in B to the subset $\tilde{A}$, trivially making the inclusion map an embedding, yielding a σ-structure on $\tilde{A}$ that allows f to be factored into a surjective homomorphism followed by a strong embedding. So it follows that condition 1 holds. It is easy to show that condition 2 holds and that this WFS is proper.

Lemma 1. *Let $(\mathscr{E}, \mathscr{M})$ be a weak factorization system in $\mathscr{A}$. The following hold:*

(a) $\mathscr{E}$ and $\mathscr{M}$ are closed under compositions;

(b) $\mathscr{E} \cap \mathscr{M} = \{isomorphisms\}$;

(c) the pushout in $\mathscr{A}$ of an $\mathscr{E}$-morphism along any morphism, if it exists, is again in $\mathscr{E}$.

Moreover, if $(\mathscr{E}, \mathscr{M})$ is proper, the following hold:

(d) $g \circ f \in \mathscr{E}$ implies $g \in \mathscr{E}$;

(e) $g \circ f \in \mathscr{M}$ implies $f \in \mathscr{M}$.

3.3.3 Lemmas for Final Theorem

In this section we present a series of lemmas which will be used in the proof of our final theorem.
Let A be a category with a proper factorization system $(\mathscr{E}, \mathscr{M})$. Denote $\mathscr{M}$-morphisms by $\rightarrowtail$ and $\mathscr{E}$-morphisms by $\twoheadrightarrow$.
Recall that pullbacks in **FinSet** admit the following explicit description: a commutative square in **FinSet** is a pullback if, and only if,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

$$\forall b \in B, \forall c \in C, \text{ if } i(b) = h(c) \text{ then } \exists! a \in A.(f(a) = b \text{ and } g(a) = c).$$

If h and i are injective then, by identifying B and C with subsets of D, the pullback A can be identified with $B \cap C$.

Lemma 2. *For any locally finite category $\mathscr{B}$ and object $n \in \mathscr{B}$, the functor $hom_{\mathscr{B}}(-, n) : \mathscr{B}^{op} \to FinSet$ maps all pushout squares in $\mathscr{B}$ to pullbacks in FinSet.*

Proof. Consider a pushout square in $\mathscr{B}$ and the corresponding pullback square in FinSet. Using the universal property of pushouts, we can verify that this mapping holds for any locally finite category.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "i"] \\ c \arrow[r, "h"'] & d \end{tikzcd} \qquad \begin{tikzcd} hom_B(d,n) \arrow[r, "-\circ i"] \arrow[d, "-\circ h"'] & hom_B(b,n) \arrow[d, "-\circ f"] \\ hom_B(c,n) \arrow[r, "-\circ g"'] & hom_B(a,n) \end{tikzcd}$$

If $\alpha \in \hom_{\mathscr{B}}(b,n)$ and $\beta \in \hom_{\mathscr{B}}(c,n)$ are such that $\alpha \circ f = \beta \circ g$ then, by the universal property of the pushout, there is a unique $\gamma \in \hom_{\mathscr{B}}(d,n)$ satisfying $\gamma \circ i = \alpha$ and $\gamma \circ h = \beta$.

□

Lemma 3. *If a functor $F : \mathscr{A}^{op} \to$ **FinSet** sends $\mathscr{E}$-pushout squares in $\mathscr{A}$ to pullbacks, then it sends $\mathscr{E}$-morphisms to injections.*

Proof. Let $e : n \twoheadrightarrow m$ be an $\mathscr{E}$-morphism in the category $\mathscr{A}$. Because e is an epimorphism, it follows directly that the square on the left-hand side below is a pushout square in $\mathscr{A}$. We can verify as follows: If we have some z with $f, g : m \to z$ such that $f \circ e = g \circ e$, then by e epi, $f = g$, thus there is a unique $f = g := h$ making the diagram commute.

$$\begin{tikzcd} n \arrow[r, "e"] \arrow[d, "e"'] & m \arrow[d, "id"] \\ m \arrow[r, "id"'] & m \end{tikzcd} \qquad \begin{tikzcd} F(m) \arrow[r, "id"] \arrow[d, "id"'] & F(m) \arrow[d, "F(e)"] \\ F(m) \arrow[r, "F(e)"'] & F(n) \end{tikzcd}$$

Since identities are $\mathscr{E}$-morphisms, then we have an $\mathscr{E}$-pushout square on the left, so the square on the right-hand side above is a pullback in ***FinSet***. *By the definition of pullbacks in* ***FinSet***, *we have that for any $m_1, m_2 \in Fm$, if $Fe(m_1) = Fe(m_2)$, then $\exists! a \in Fm$ such that $id(a) = m_1$ and $id(a) = m_2$, ie. $m_1 = m_2$, so it follows that Fe is injective.* □

Generic and Degenerate Elements. Major step in proof of Lovasz' theorem: Showing that if $|\hom_{\Sigma_f}(C, A)| = |\hom_{\Sigma_f}(C, B)|$ for all $C \in \Sigma_f$, then $|\mathrm{inj}_{\Sigma_f}(C, A)| = |\mathrm{inj}_{\Sigma_f}(C, B)|$ for all $C \in \Sigma_f$, where $\mathrm{inj}_{\Sigma_f}(C, A)$ denotes the set of all injective homomorphisms $C \to A$. In our generalized setting, we show something similar but with $\mathscr{M}$-morphisms.

The intuition behind the next definition is that a σ-homomorphism $f : C \to A$ is non-injective when there exists a surjective non-injective σ-homomorphism $e : C \twoheadrightarrow C'$ and a σ-homomorphism $g : C' \to A$ such that $f = g \circ e$. We can restate this in terms of the hom-functor $E := \hom_{\Sigma_f}(-, A)$ as: $f \in E(C)$ is non-injective if and only if $f = E(e)(g)$ for some onto, non-injective $e : C \twoheadrightarrow C'$ and $g \in E(C')$.

Definition 15. *A strict quotient in $\mathscr{A}$ is an $\mathscr{E}$-morphism that is not an isomorphism. (Equivalently, by Lemma 1(b), a strict quotient is an arrow in $\mathscr{E} \setminus \mathscr{M}$.)*

For a functor $E : \mathscr{A}^{op} \to FinSet$, we say that $s \in E(k)$ is degenerate if there exists a strict quotient $f : k \twoheadrightarrow l$ and $t \in E(l)$ such that $E(f)(t) = s$. Otherwise, s is called generic. The set of generic elements in $E(k)$ is denoted by $E^{gen}[k]$.

The next lemma formalizes how this definition aligns with our intuition for hom-functors on $\mathscr{A}$.

Lemma 4. *Let $E = hom_{\mathscr{A}}(-, n)$ for some $n \in \mathscr{A}$. For any $k \in \mathscr{A}$, $E^{gen}[k]$ is the set of all $\mathscr{M}$-morphisms $k \to n$.*
Proof. Let f be an arbitrary element of $E(k) = hom_{\mathscr{A}}(k, n)$ and assume f is generic. Take its $(\mathscr{E}, \mathscr{M})$ factorization:

$$k \xrightarrow[g]{} \!\!\!\!\rightarrow l \rightarrowtail^{h} n$$

Then $E(g)(h) = f$, since f is generic, by definition g can not be a strict quotient, so g must be an $\mathscr{M}$-morphism. By Lemma 1(a), $f = h \circ g$ is also an $\mathscr{M}$-morphism.

Conversely, suppose f is an $\mathscr{M}$-morphism. Pick an $\mathscr{E}$-morphism $g : k \twoheadrightarrow l$ and $h \in E(l)$ such that $E(g)(h) = f$, i.e., $h \circ g = f$. By Lemma 1(e), g is an $\mathscr{M}$-morphism, so it is not a strict quotient. Hence, f is generic.

The following is the main technical lemma of this section, relying on an application of the inclusion-exclusion principle.

Lemma 5. *Assume $\mathscr{A}$ has pushouts, and let E and F be functors $\mathscr{A}^{op} \to FinSet$ sending $\mathscr{E}$-pushout squares in $\mathscr{A}$ to pullbacks. If $E(k) \cong F(k)$ for all $k \in \mathscr{A}$, then $E^{gen}[k] \cong F^{gen}[k]$ for all $k \in \mathscr{A}$.*

Proof. *Let E, F be as in the statement and suppose $E(k) \cong F(k)$ for all $k \in \mathscr{A}$. We show $E^{gen}[k] \cong F^{gen}[k]$ by proving a bijection between degenerate elements in $E(k)$ and $F(k)$, denoted $E(|k|), F(|k|)$.*

Observe that $E(|k|)$ coincides with

$$\bigcup\{Im(E(f)) \mid f : k \twoheadrightarrow l \text{ is a strict quotient in } \mathscr{A}\} \subseteq E(k),$$

and similarly for $F(|k|)$. Since $E(k)$ and $F(k)$ are finite, then we have a finite set S of strict quotients of k such that

$$E(|k|) = \bigcup\{Im(E(f)) \mid f \in S\}, \quad F(|k|) = \bigcup\{Im(F(f)) \mid f \in S\}.$$

Inclusion-Exclusion Principle. The inclusion-exclusion principle provides a way to compute the size of a union of finite sets by accounting for overlaps between them. Given a finite collection of sets $A_1, A_2, \ldots, A_n$, the size of the union $|A_1 \cup A_2 \cup \cdots \cup A_n|$ is given by

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|.$$

This formula alternates between adding and subtracting intersections of increasing size to ensure each element is counted exactly once. For example, we add the sizes of individual sets, subtract pairwise intersections, add triple intersections, and so on, until we reach the intersection of all sets.

Applying this to our case, by the inclusion-exclusion principle,

$$|E(|k|)| = \sum_{J \subseteq S, J \neq \emptyset} (-1)^{|J|+1} \left|\bigcap_{f \in J} Im(E(f))\right|,$$

and similarly for $|F(|k|)|$. Thus, it suffices to show that $\bigcap_{f \in J} Im(E(f)) \cong \bigcap_{f \in J} Im(F(f))$ for any non-empty $J \subseteq S$.

Fix such J and consider the wide pushout in $\mathscr{A}$ of the strict quotients in J, as shown below:

$$\begin{tikzcd}
k \arrow[rrr, two heads] \arrow[drr, two heads] \arrow[ddr, two heads] \arrow[ddd, two heads] & & & l_u \arrow[ddd, two heads] \\
& & l_3 \arrow[ddr, two heads] \arrow[ur, phantom, "\cdots"] & \\
& l_2 \arrow[drr, two heads] & & \\
l_1 \arrow[rrr, two heads] & & & p
\end{tikzcd}
\qquad
\begin{tikzcd}
E(k) & & & E(l_u) \arrow[lll, hook'] \\
& & E(l_3) \arrow[ull, hook'] \arrow[ur, phantom, "\cdots"] & \\
& E(l_2) \arrow[uul, hook'] & & \\
E(l_1) \arrow[uuu, hook'] & & & E(p) \arrow[lll, hook'] \arrow[ull, hook'] \arrow[uul, hook'] \arrow[uuu, hook']
\end{tikzcd}$$

By Lemma 1(c), all arrows in the left diagram are $\mathscr{E}$-morphisms. Thus, the diagram on the right is a pullback diagram of injections in FinSet, so by definition of pullbacks in Set, $E(p) \cong \bigcap_{f \in J} Im(E(f))$. Similarly, $F(p) \cong \bigcap_{f \in J} Im(F(f))$, so $|\bigcap_{f \in J} Im(E(f))| = |\bigcap_{f \in J} Im(F(f))|$. This completes the proof.

3.3.4 Proof of the Generalized Lovász Theorem

We now proceed with the main result, proving that Σ is combinatorial.

Theorem 6. *Let Σ be a locally finite category with pushouts and a proper factorization system $(\mathscr{E}, \mathscr{M})$. Then Σ is combinatorial.*

Proof. Fix arbitrary objects $m, n \in \mathscr{A}$. For the non-trivial direction, assume $|\hom_{\mathscr{A}}(k, m)| = |\hom_{\mathscr{A}}(k, n)|$ for all $k \in \mathscr{A}$. Let E and F denote, respectively, the functors $\hom_{\mathscr{A}}(-, m) : \mathscr{A}^{\mathrm{op}} \to \mathrm{FinSet}$ and $\hom_{\mathscr{A}}(-, n) : \mathscr{A}^{\mathrm{op}} \to \mathrm{FinSet}$. These functors send $\mathscr{E}$-pushout squares in $\mathscr{A}$ to pullbacks by Lemma 2. Thus, by Lemma 5, $E[\![k]\!] \cong F[\![k]\!]$ for all $k \in \mathscr{A}$. According to Lemma 4, there is a bijection

$$\{\mathscr{M}\text{-morphisms } k \to m\} \cong \{\mathscr{M}\text{-morphisms } k \to n\}.$$

Setting $k = m$, the existence of an $\mathscr{M}$-morphism $m \to m$ (namely the identity) entails the existence of an $\mathscr{M}$-morphism $i : m \to n$. Similarly, there exists an $\mathscr{M}$-morphism $j : n \to m$. A standard argument then shows that $m \cong n$.

We briefly sketch a proof. The set L of all $\mathscr{M}$-morphisms $m \to m$ is a monoid with respect to composition and contains $j \circ i$. Since all $\mathscr{M}$-morphisms are monos, L satisfies the left cancellation law $ab = ac \Rightarrow b = c$. But every finite monoid satisfying the left cancellation law is a group, hence $j \circ i$ has an inverse. It follows from lemma 1(b),(d) that j is an isomorphism. □

4 Conclusion

In these lectures, we connected category theory with finite model theory through the concept of game comonads. We began by introducing the Ehrenfeucht-Fraisse (EF) game and demonstrated how it can be represented as a comonad within the category Σ of relational structures. This representation allowed us to establish a direct correspondence between morphisms in the coKleisli category and Duplicator's winning strategies in the k-round existential EF game.

We explored the role of coalgebras for the EF comonad, showing that the coalgebra number of a structure aligns with its tree depth.

We then extended Lovazs' homomorphism counting theorem to a categorical framework. By proving that locally finite categories with pushouts and a proper factorization system are combinatorial, we generalized the classical result to a broader context. This extension underscores the power of category-theoretic approaches in capturing fundamental combinatorial characteristics.

Overall, these lectures illustrate how game comonads serve as a robust tool for bridging structural semantics and combinatorial complexity in finite model theory. This framework not only provides a deeper understanding of model comparison games but also offers a unified approach to generalizing important combinatorial theorems within category theory.

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